

Neutrix Calculus and Finite Quantum Field Theory

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Abstract

In general, quantum field theories (QFT) require regularizations and *infinite* renormalizations due to ultraviolet divergences in their loop calculations. Furthermore, perturbation series in theories like QED are not convergent series, but are asymptotic series. We apply neutrix calculus, developed in connection with asymptotic series and divergent integrals, to QFT, obtaining *finite* renormalizations. While none of the physically measurable results in renormalizable QFT is changed, quantum gravity is rendered more manageable in the neutrix framework.

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The procedure of regularization and renormalization is a big step forward in making sense of the infinities that one encounters in calculating perturbative series in quantum field theories. The result is a phenomenal success. For example, Quantum Electrodynamics (QED), the paradigm of relativistic quantum field theories, suitably regularized and renormalized, is arguably the most accurate theory ever devised by mankind. Yet in spite of the impressive phenomenological successes, the specter of infinite renormalizations has convinced many, including such eminent physicists as Dirac and Schwinger, that we should seek a better mathematical and/or physical foundation for quantum field theory, without simultaneously tearing down the towering edifice we have built on the existing one. In another development, Dyson [1] has shown that the series as defined by the Feynman rules in QED is not a convergent series and has suggested that it is instead an asymptotic series in the fine structure constant α , i.e., in the number of internal integrals (for given outside lines). In this paper, we propose to apply neutrix calculus, in conjunction with Hadamard integrals, developed by J.G. van der Corput [2] in connection with asymptotic series and divergent integrals, to quantum field theories in general, and QED in particular, to obtain finite results for the coefficients in the perturbation series. (A more detailed discussion[3] will appear elsewhere.) The replacement of regular integrals by Hadamard integrals in quantum field theory appears to make good mathematical sense, as van der Corput observed that Hadamard integrals are the proper tool to calculate the coefficients of an asymptotic series. (Actually Hadamard integrals work equally well for convergent series.)

We begin by recalling the definition of asymptotic series[4]. The series $f(x) = a_0 + a_1(x - b) + a_2(x - b)^2 + \dots$ for finite b is an asymptotic series if and only if there exists an $n_0 > 0$, such that for $n > n_0$,

$$\lim_{x \rightarrow b} \frac{1}{(x - b)^n} |f(x) - a_0 - a_1(x - b) - \dots - a_n(x - b)^n| = 0, \quad (1)$$

with the remnant being at most $\sim (x - b)^{n+1}$.

Next, following van der Corput [2], we define a neutrix as a class of negligible functions defined in a domain, which satisfy the following two conditions: (1) the neutrix is an additive group; (2) it does not contain any constant except 0. Let us illustrate the concept with the following example considered by Hadamard: for s real,

$$\int_{\xi}^2 x^{s-1} dx = \begin{cases} s^{-1}2^s - s^{-1}\xi^s & \text{for } s \neq 0 \\ \log 2 - \log \xi & \text{for } s = 0 \end{cases} \quad (2)$$

For $s > 0$, the integral converges even as $\xi \rightarrow 0$. For $s \leq 0$, Hadamard neglects ξ^s/s and $\log \xi$ as $\xi \rightarrow 0$. Here we have a neutrix which we will call $N(0)$, consisting of functions $\nu(\xi) = \epsilon(\xi) + c_1 \xi^s + c_2 \log \xi$, where $\epsilon(\xi) \rightarrow 0$ as $\xi \rightarrow 0$, and where c_1 , c_2 , and s are arbitrary constants. This results in writing

$$\int_{N(0)}^2 x^{s-1} dx = \begin{cases} s^{-1} 2^s & \text{for } s \neq 0 \\ \log 2 & \text{for } s = 0 \end{cases} \quad (3)$$

Note the analytic extension in the complex s plane of the answer for $\text{Re } s > 0$ to the entire complex plane with the exclusion of $s = 0$.

Before applying neutrices to QED, we need to consider the generalized Hadamard neutrix H_a defined to contain the negligible functions

$$\nu(\xi) = U(\xi) + \epsilon(\xi), \quad (4)$$

where $\epsilon(\xi) \rightarrow 0$ as $\xi \rightarrow a$. Each of the functions $U(\xi)$ is defined by an asymptotic series based on a :

$$U(\xi) \sim \sum_{h=0}^{\infty} \chi_h (\xi - a)^{\Psi_h} \log^{k_h}(\xi - a). \quad (5)$$

Here χ_h , Ψ_h and integers $k_h \geq 0$ are independent of ξ , $\text{Re } \Psi_h \rightarrow \infty$ as $h \rightarrow \infty$, and $\log^{k_h}(\xi)$ stands for $(\log(\xi))^{k_h}$. An example is provided by

$$\int_{H_a}^b (z - a)^{-1} \log^k(z - a) dz = (k + 1)^{-1} \log^{k+1}(b - a). \quad (6)$$

Similarly, we can define the Hadamard neutrix H_{∞} by Eq. (4) where now $\epsilon(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$ and the function $U(\xi)$ has a Hadamard development in powers of ξ^{-1} in its asymptotic series:

$$U(\xi) \sim \sum_{h=0}^{\infty} \chi_h \xi^{\Psi_h} \log^{k_h} \xi, \quad (7)$$

where $\text{Re } \Psi_h \rightarrow -\infty$ as $h \rightarrow \infty$.

We can now demonstrate a very valuable property of the Hadamard neutrix. Recall that in the theory of distributions developed by Schwartz, generalized functions usually cannot be multiplied. Consider, for example, the one-dimensional Dirac delta function multiplying itself $\delta(x) \times \delta(x)$. This product is not mathematically meaningful because its Fourier transform diverges:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} 1 \times 1 \longrightarrow \infty, \quad (8)$$

where we have used the convolution rule in Fourier transform and have noted that the Fourier transform of the Dirac delta function is 1. In contrast, the Hadamard method does allow multiplication for a wide class of distributions. For the example of $\delta(x) \times \delta(x)$, the Hadamard-neutralized Fourier transform of the product

$$\int_{H-\infty}^{H\infty} \frac{dk}{2\pi} 1 \times 1 = 0, \quad (9)$$

yields

$$\delta(x) \times \delta(x) = 0, \quad (10)$$

a mathematically meaningful (though somewhat counter-intuitive) result!

In doing quantum field theory in configuration space, we multiply operator-valued distributions of quantum fields. Or, in a slightly different interpretation, we multiply singular functions such as Feynman propagators. As we will see, the use of neutrix calculus allows one to put these products on a mathematically sound basis. Let us now generalize the above discussion for 1-dimensional Dirac delta functions to the case of $(3 + 1)$ -dimensional Feynman propagators

$$\Delta_+(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip \cdot x}}{p^2 + m^2 - i\epsilon} \quad (11)$$

$$= \frac{1}{4\pi} \delta(x^2) - \frac{m}{8\pi\sqrt{-x^2 - i\epsilon}} H_1^{(2)}(m\sqrt{-x^2 - i\epsilon}), \quad (12)$$

where $H^{(2)}$ is the Hankel function of the second kind and we use the $(+++)$ metric. The Fourier transform of $\Delta_+(x) \times \Delta_+(x)$ (which appears in certain quantum loop calculations) is given by

$$\int d^4x e^{-ip \cdot x} \Delta_+(x) \Delta_+(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(p - k)^2 + m^2 - i\epsilon} \quad (13)$$

$$= \frac{i}{4(2\pi)^2} D - \frac{i}{4(2\pi)^2} \int_0^1 dz \log \left(1 + \frac{p^2}{m^2} z(1 - z) \right), \quad (14)$$

where

$$\begin{aligned} D &= \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2 + m^2)^2} \\ &= \int_0^\infty \frac{k^2 dk^2}{(k^2 + m^2)^2}, \end{aligned} \quad (15)$$

with the second expression of D obtained after a Wick rotation. But D is logarithmically divergent. Hence $\Delta_+(x) \times \Delta_+(x)$ is not mathematically well defined. Let us now see how the

Hadamard-van der Corput method gives mathematical meaning to this product. Obviously, it is in the calculation of the logarithmically divergent D where we apply neutrix calculus. Introducing dimensionless variable $q = k^2/m^2$, we bring in H_∞ to write D as

$$D = \int_0^{H_\infty} \frac{q dq}{(q+1)^2} = -1, \quad (16)$$

where we have recalled that, for $q \rightarrow \infty$, $\log q$ is negligible in the Hadamard neutrix H_∞ . It follows that, in the neutralized version, $\Delta_+(x) \times \Delta_+(x) \sim \delta^{(4)}(x) + \text{regular part}$ (where $\delta^{(4)}(x)$ is the 4-dimensional Dirac delta function), a much more mathematically palatable object.

As the first example in the application of neutrices to QED, let us consider the one-loop contribution to the electron's self energy

$$\Sigma(p) = -ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\gamma_\mu [-\gamma \cdot (p-k) + m] \gamma^\mu}{[k^2 + \lambda^2][(p-k)^2 + m^2]}, \quad (17)$$

where m is the electron bare mass and we have given the photon a fictitious mass λ to regularize infrared divergences. Expanding $\Sigma(p)$ about $\gamma \cdot p = -m$,

$$\Sigma(p) = \mathcal{A} + \mathcal{B}(\gamma \cdot p + m) + \mathcal{R}, \quad (18)$$

one finds (cf. results found in Ref.[5])

$$\mathcal{A} = -\frac{\alpha}{2\pi} m \left(\frac{3}{2} D + \frac{9}{4} \right), \quad (19)$$

$$\mathcal{B} = -\frac{\alpha}{4\pi} \left(D - 4 \int_{\frac{\lambda}{m}}^1 \frac{dx}{x} + \frac{11}{2} \right), \quad (20)$$

where $\alpha = e^2/4\pi$ is the fine structure constant and D is given by Eq. (15) and Eq. (16) in the pre-neutralized and neutralized forms respectively. We note that \mathcal{R} , the last piece of $\Sigma(p)$ in Eq. (18), is finite. Mass renormalization and wavefunction renormalization are given by $m_{ren} = m - \mathcal{A}$ and $\psi_{ren} = Z_2^{-1/2} \psi$ respectively with $Z_2^{-1} = 1 - \mathcal{B}$. Now, since $D = -1$ is finite, it is abundantly clear that the renormalizations are *finite* in the framework of neutrix calculus. There is no need for a separate discussion of the electron vertex function renormalization constant Z_1 due to the Ward identity $Z_1 = Z_2$.

The one-loop contribution to vacuum polarization is given by

$$\Pi_{\mu\nu}(k) = ie^2 \int \frac{d^4 p}{(2\pi)^4} Tr \left(\gamma_\mu \frac{1}{\gamma \cdot (p + \frac{k}{2}) + m} \gamma_\nu \frac{1}{\gamma \cdot (p - \frac{k}{2}) + m} \right). \quad (21)$$

A standard calculation[5] shows that $\Pi_{\mu\nu}$ takes on the form

$$\Pi_{\mu\nu} = \delta m^2 \eta_{\mu\nu} + (k^2 \eta_{\mu\nu} - k_\mu k_\nu) \Pi(k^2), \quad (22)$$

where $\eta_{\mu\nu}$ is the flat metric $(+++-)$,

$$\delta m^2 = \frac{\alpha}{2\pi} (m^2 D + D'), \quad (23)$$

and

$$\Pi(k^2) = -\frac{\alpha}{3\pi} (D + \frac{5}{6}) + \frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log \left(1 + \frac{k^2}{m^2} x(1-x) \right), \quad (24)$$

with

$$D' = \frac{1}{i\pi^2} \int \frac{d^4 p}{p^2 + m^2}, \quad (25)$$

and D given by Eq. (15). Just as D is rendered finite upon invoking neutrix calculus (see Eq. (16)), so is D' :

$$D' = m^2 \int_0^{H_\infty} \frac{q dq}{q+1} = 0, \quad (26)$$

since both q and $\log q$, for $q \rightarrow \infty$, are negligible in H_∞ . Thus neutrix calculus yields a finite renormalization for both the photon mass and the photon wavefunction $A_{ren}^\mu = Z_3^{-1/2} A^\mu$ (and consequently also for charge $e_{ren} = Z_3^{1/2} e$) where $Z_3^{-1} = 1 - \Pi(0)$. In electron-electron scattering by the exchange of a photon with energy-momentum k , vacuum polarization effects effectively replace e^2 by $e^2/(1 - \Pi(k^2))$, i.e.,

$$\begin{aligned} e^2 \rightarrow e_{eff}^2 &= \frac{e^2}{1 - \Pi(k^2)} \\ &= \frac{e_{ren}^2}{Z_3(1 - \Pi(k^2))} \\ &= \frac{e_{ren}^2}{1 - (\Pi(k^2) - \Pi(0))}. \end{aligned} \quad (27)$$

Eq. (24) can be used, for $k^2 \gg m^2$, to show that

$$\alpha_{eff}(k^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log \left(\frac{k^2}{\exp(5/3) m^2} \right)}. \quad (28)$$

Thus we have obtained the correct running of the coupling[6] with energy-momentum in the framework of neutrices. In fact, the *only* effect of neutrix calculus, when applied to QED (and other renormalizable theories), is to convert *infinite* renormalizations (obtained without using neutrix calculus) to mathematically well-defined *finite* renormalizations. As

far as we can tell, *all* (finite) physically observable results of QED are recovered. In passing we mention that the use of neutralized integrals does not affect the results of axial triangle anomalies.

As shown by the appearance of photon mass in the above discussion of vacuum polarization, the application of neutrix calculus to the energy-momentum cutoff regularization, though straightforward and natural, is ill suited for more complicated theories like those involving Yang-Mills fields. For those theories, one should use other more convenient regularization schemes. It is amusing to note that already in 1961 van der Corput suggested that, instead of finding the appropriate neutrices, one can continue analytically in any variable (presumably including the dimension of integrations) contained in the problem of tackling apparent divergences to calculate the coefficients of the corresponding asymptotic series. In hind sight, one recognizes that this was the approach taken by 't Hooft and Veltman who spearheaded the use of dimensional regularizations[6]. Let us now explore using neutrix calculus in conjunction with the dimensional regularization scheme. In that case, negligible functions will include $1/\epsilon$ where $\epsilon = 4 - n$ is the deviation of spacetime dimensions from 4. In the one-loop calculations for QED, the internal energy-momentum integration is now over n dimensions. The forms of $\Sigma(p)$ and $\Pi_{\mu\nu}$ remain the same as given by Eqs. (18) and (22), but now with $\delta m^2 = 0$. Using the approximation for the gamma function, $\Gamma(\epsilon) = \epsilon^{-1} - \gamma + \mathcal{O}(\epsilon)$, where $\gamma \simeq .577$ is the Euler-Mascheroni constant, and the approximation $f^\epsilon \simeq 1 + \epsilon \log f$, for $\epsilon \ll 1$, one finds

$$\begin{aligned}\mathcal{A} &= \frac{\alpha m}{4\pi} [3(\gamma - \log 4\pi) + 1] + \frac{\alpha}{2\pi} m \int_0^1 dx (1+x) \log D_0, \\ \mathcal{B} &= \frac{\alpha}{4\pi} [1 + \gamma - \log 4\pi] + \frac{\alpha m^2}{\pi} \int_0^1 dx \frac{x(1-x^2)}{m^2 x^2 + \lambda^2(1-x)} + \frac{\alpha}{2\pi} \int_0^1 dx (1-x) \log D_0, \quad (29)\end{aligned}$$

where $D_0 = m^2 x^2 + \lambda^2(1-x)$, and

$$\begin{aligned}\Pi(k^2) &= \frac{\alpha}{3\pi} [\gamma - \log 4\pi] + \frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log[m^2 + x(1-x)k^2], \\ \frac{1}{Z_3} &= 1 - \frac{\alpha}{3\pi} [\gamma - \log 4\pi] - \frac{\alpha}{3\pi} \log m^2. \quad (30)\end{aligned}$$

By design, the generalized neutrix calculus renders all the renormalizations *finite*. Again, *all* physically measurable results of QED appear to be recovered. In this article we have explicitly considered QED to one-loop only. But we expect that higher-loop calculations can be handled in the same way according to neutrix calculus. It will be interesting to

see explicitly whether neutrix calculus, applied to higher-loop calculations, can provide new insights in the issue of overlapping divergences.

In the framework of quantum field theory for the four fundamental forces, the divergence problem is particularly severe for quantum gravity. Using dimensional regularization, 'tHooft and Veltman[7] found that pure gravity is one-loop renormalizable, but in the presence of a scalar field, renormalization was lost. For the latter case, they found that the counterterm evaluated on the mass shell is given by $\sim \epsilon^{-1} \sqrt{g} R^2$ with R being the Ricci scalar. Similar results for the cases of Maxwell fields and Dirac fields etc (supplementing the Einstein field) were obtained[8]. It is natural to inquire whether the application of neutrix calculus could improve the situation. The result is that now essentially the divergent ϵ^{-1} factor is replaced by $-\gamma + \text{constant}$.

It has not escaped our notice that neutrix calculus may ameliorate the hierarchy problem in particle physics. The hierarchy problem is due to the fact that the Higgs scalar self-energies diverge quadratically, leading to a stability problem in the standard model of particle physics. But neutrix calculus treats quadratic divergences no different from logarithmic divergences, since both divergences belong to (the negligible functions of) the neutrix. Neutrix calculus may also ameliorate the cosmological constant problem in quantum gravity. The cosmological constant problem can be traced to the quartic divergences in zero-point fluctuations from all quantum fields. But again, neutrix calculus treats quartic divergences no different from logarithmic divergences. Indeed, for a theory of gravitation with a cosmological constant term, the cosmological constant receives at most a finite renormalization from the quantum loops in the framework of neutrix calculus.

We conclude with a comment on what neutrix calculus means to the general question of renormalizability of a theory. We recall that a theory is renormalizable if, in loop calculations, the counterterms vanish or if they are proportional to terms in the original Lagrangian (the usual renormalization through rescaling). It is still renormalizable if, to all loops, the counterterms are of a new form, but only a finite number of such terms exist. By this standard, neutrix calculus does not change the renormalizability of a theory, since it merely changes potentially infinite renormalizations to finite renormalizations. On the other hand, non-renormalizable terms, i.e., terms with positive superficial degree of divergence, are tolerated in neutralized quantum field theory. In a sense it is a pity that we have lost renormalizability as a physical restrictive criterion in the choice of sensible theories. However, we

believe that this is actually not as big a loss as it may first appear. Quite likely, all realistic theories now in our possession are actually effective field theories.[9, 10] They appear to be renormalizable field theories because, at energies now accessible, or more correctly, at sufficiently low energies, all the non-renormalizable interactions are highly suppressed. By tolerating non-renormalizable terms, neutrix calculus has freed us from the past dogmatic and rigid requirement of renormalizability. (Having said that, given a choice between renormalizable field theories and effective field theories, we still prefer the former to the latter because of the former's compactness and predictive power. But the point is that both types of theories can be accommodated in the framework of neutrix calculus.) Furthermore, if the application of neutrix calculus to loop calculations results in a term of a new form (like the Pauli term in QED) that is finite, then we have a prediction which, in principle, can be checked against experiments to confirm or invalidate the theory in question. For the latter case, we will have to modify the theory by including a term of that form in the Lagrangian, making the parameter associated with the new term an adjustable parameter rather than one that is predicted by the theory. This loss of predictive power is again not as big a loss as one may dread.

Lastly we should emphasize that, for renormalizable theories as well as non-renormalizable theories (like quantum gravity?), neutrix calculus is a useful tool to the extent that it is relevant for asymptotic series and lessens the divergence of the theories. Based on our study so far, we tentatively conclude that neutrix calculus has banished infinities from quantum field theory, rendering perturbative quantum field theory mathematically meaningful.

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